JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 91 (1998) 179–190

On certain generalized incomplete gamma functions

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Abstract

Recently, Chaudhry and Zubair have introduced a generalized incomplete gamma function $\Gamma(\nu,x;z)$ which reduces to the incomplete gamma function $\Gamma(\nu,x)$ when its variable z vanishes. We show that $\Gamma(\nu,x;z)$ may be written essentially as a single Kampé de Fériet function which in turn may be expressed as a linear combination of two incomplete Weber integrals. Then by using properties of the latter integrals we deduce additional representations for $\Gamma(\nu,x;z)$. In particular, we show that $\Gamma(\nu,x;z)$ is essentially completely determined by a finite number of modified Bessel functions for all $\nu \neq 0$ provided we know the values of the two incomplete Weber integrals when $0 < \operatorname{Re} \nu \leqslant 1$. When $\nu = 0$ we derive connections between the generalized incomplete gamma function and incomplete Lipschitz–Hankel integrals, and indicate that there exist connections with other special functions.

AMS classification: 33B20; 33C10; 33C70

Keywords: Generalized incomplete gamma functions; Bessel functions; Kampé de Fériet functions

1. Introduction

Recently, Chaudhry and Zubair have introduced a generalized incomplete gamma function defined for complex parameters ν,x and complex variable z by

$$\Gamma(\nu,x;z) \equiv \int_x^\infty t^{\nu-1} e^{-t-z/t}\, dt. \tag{1.1}$$

This function may be employed to express solutions to various important problems in applied mathematics and statistics (see [4] for further details and references). It is the purpose of the present

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[1] Research supported by the Office of Naval Research.

0377-0427/98/$19.00
PII S0377-0427(98)00031-4

investigation to show that there exist connections between the generalized function defined by Eq. (1.1) and other known special functions, e.g. Kampé de Fériet (KdF) functions and incomplete Weber integrals. Indeed, we shall show in Section 6 that $\Gamma(\nu,x;z)$ is essentially completely determined by a finite number of modified Bessel functions for all $\nu \neq 0$ provided we know the values of two incomplete Weber integrals when $0 < \mathrm{Re}\ \nu \leqslant 1$. In particular, for n an integer $\Gamma(\frac{1}{2}+n,x;z)$ may be written essentially as a finite number of modified Bessel functions of half odd integer order; this has already been shown in [3, Eq. (6.1)] only for $n=-1,0,1,\ldots$. It will be convenient, in what follows, to call the parameter ν the order of the generalized function $\Gamma(\nu,x;z)$.

From [4, Eqs. (4), (12)] it is easy to obtain the useful and important functional characterization:

$$K_\nu(2\sqrt{z}) = \tfrac{1}{2}z^{\nu/2}\Gamma(-\nu,x^{-1}\sqrt{z};z) + \tfrac{1}{2}z^{-\nu/2}\Gamma(\nu,x\sqrt{z};z), \tag{1.2}$$

where $K_\nu(z)$ is the Bessel function of imaginary argument or Macdonald function. Obviously, Eq. (1.2) immediately gives

$$\Gamma(0,\sqrt{z};z) = K_0(2\sqrt{z}). \tag{1.3}$$

Representations for $\Gamma(0,x;z)$ for arbitrary nonvanishing x shall be derived later (see Eqs. (3.3), (3.5) with $n=0$, and (7.5)).

2. Representation for $\Gamma(\nu,x;z)$ in terms of a KdF function

We shall need the following preliminary lemma pertaining to the incomplete gamma function. For $p=0,1,2,\ldots$

$$\Gamma(\nu-p,x) = \frac{(-1)^p}{\nu(1-\nu)_p}\left[\Gamma(1+\nu,x) - e^{-x}x^\nu \sum_{k=0}^{p}(-\nu)_k\left(-\frac{1}{x}\right)^k\right]. \tag{2.1a}$$

This result may be obtained by replacing ν by $\nu-p$ in the well-known identity

$$\Gamma(\nu,x) = \Gamma(\nu) - \frac{e^{-x}x^\nu}{\nu}\,{}_1F_1[1;\ 1+\nu;\ x].$$

Then by writing the latter confluent function as a hypergeometric sum, employing elementary manipulations involving the adjustment of summation indices and properties of the gamma function, we deduce Eq. (2.1a). The computation just described although straightforward is somewhat tedious; we have therefore omitted it since we shall derive Eq. (2.1a) quickly in another way in Section 5 (cf. Eq. (5.7)). A similar computation also employing the later identity with ν replaced by $\nu+p+1$ yields the result

$$\Gamma(\nu+p+1,x) = \nu(1+\nu)_p\left[\Gamma(\nu,x) + \frac{e^{-x}x^\nu}{\nu}\sum_{k=0}^{p}\frac{x^k}{(1+\nu)_k}\right], \tag{2.1b}$$

where p is a nonnegative integer. Eq. (2.1a) is an obvious generalization of the well-known special case

$$\Gamma(-n,x)=\frac{(-1)^n}{n!}\left[\Gamma(0,x)-\frac{\mathrm{e}^{-x}}{x}\sum_{k=0}^{n-1}k!\left(-\frac{1}{x}\right)^k\right], \tag{2.2}$$

where $n=1,2,3,\ldots$.

We adapt throughout the convention that when the upper limit of a summation is less than the initial value of the lower limit, then the summation vanishes. Thus, Eqs. (2.1) and (2.2), respectively, become identities when $p=-1$ and $n=0$. We note that by setting $p=0$ in Eqs. (2.1), we retrieve the familiar functional relation for the incomplete gamma function:

$$\Gamma(v+1,x)=v\Gamma(v,x)+\mathrm{e}^{-x}x^{v}. \tag{2.3}$$

By writing $\exp(-z/t)$ as a Maclaurin series, Eq. (1.1) may be written as

$$\Gamma(v,x;z)=\sum_{m=0}^{\infty}\frac{(-z)^m}{m!}\Gamma(v-m,x) \tag{2.4}$$

which as already pointed out in [3] is just the Maclaurin expansion of $\Gamma(v,x;z)$. Now, by using Eq. (2.1a) to compute $\Gamma(v-m,x)$, we see that

$$\begin{aligned}\Gamma(-v,x;z) =&\frac{1}{v}\left(\mathrm{e}^{-x}x^{-v}\sum_{m=0}^{\infty}\frac{z^m}{(1+v)_m\,m!}\sum_{k=0}^{m}(v)_k\left(-\frac{1}{x}\right)^k\right.\\ &\left.-\Gamma(1-v,x)\,{}_0F_1[-;1+v;z]\right).\end{aligned} \tag{2.5}$$

Finally, by rearranging the double series in Eq. (2.5) (cf. Eq. (3.4)) and noting that the modified Bessel function $I_v(z)$ may be written as

$$I_v(z)=\frac{(z/2)^v}{\Gamma(1+v)}\,{}_0F_1[-;1+v;z^2/4] \tag{2.6}$$

we deduce

$$\begin{aligned}\Gamma\left(-v,\frac{1}{x};\frac{z^2}{4}\right) =&\frac{x^v\mathrm{e}^{-1/x}}{v}F^{0:2;1}_{2:0;0}\left[\begin{matrix}\text{———}: 1,v;\ 1;\\ 1,1+v: \text{—};\text{—};\end{matrix}\ -x\frac{z^2}{4},\ \frac{z^2}{4}\right]\\ &-\Gamma\left(1-v,\frac{1}{x}\right)\Gamma(v)\left(\frac{2}{z}\right)^v I_v(z),\end{aligned} \tag{2.7}$$

where $v\neq 0,-1,-2,\ldots$.

Since

$$\begin{aligned}&F^{0:2;1}_{2:0;0}\left[\begin{matrix}\text{———}: 1,v;\ 1;\\ 1,1+v: \text{—};\text{—};\end{matrix}\ -x\frac{z^2}{4},\ \frac{z^2}{4}\right]\\ &={}_0F_1[-;\ 1+v;\ z^2/4]-\frac{v}{1+v}\left(x\frac{z^2}{4}\right)F^{0:2;1}_{2:0;0}\left[\begin{matrix}\text{———}: 1,1+v;\ 1;\\ 2,2+v: \text{———};\text{—};\end{matrix}\ -x\frac{z^2}{4},\ \frac{z^2}{4}\right]\end{aligned}$$

Eqs. (2.3), (2.6) and (2.7) yield

$$\Gamma\left(-\nu,\frac{1}{x};\frac{z^2}{4}\right)=\Gamma\left(-\nu,\frac{1}{x}\right)\Gamma(1+\nu)\left(\frac{2}{z}\right)^{\nu}I_\nu(z)$$

$$-\frac{z^2}{4}\frac{x^{1+\nu}}{1+\nu}\mathrm{e}^{-1/x}F^{0:2;1}_{2:0;0}\left[\begin{matrix}\text{——}: 1,1+\nu;\ 1;\\ 2,2+\nu:\text{——};\ \text{—};\end{matrix}\ -x\frac{z^2}{4},\ \frac{z^2}{4}\right], \tag{2.8}$$

where now $\nu\neq -1,\ -2,\ -3,\dots$.

3. The generalized function for nonnegative integer order n

Eqs. (2.7) and (2.8) are obviously of no use when ν is a negative integer. However, a procedure along the lines of the derivation of Eq. (2.7) mutatus mutandis, but now utilizing Eq. (2.2) (instead of Eq. (2.1a)) provides the result

$$\Gamma(n,x;z)=\frac{(-z)^n}{n!}\left\{\sum_{k=1}^{n}\frac{(-n)_k}{z^k}\Gamma(k,x)+{}_0F_1[-;\ n+1;\ z]\Gamma(0,x)\right.$$

$$\left.-\frac{\mathrm{e}^{-x}}{x}\sum_{m=1}^{\infty}\frac{z^m}{(n+1)_m m!}\sum_{k=0}^{m-1}k!\left(-\frac{1}{x}\right)^k\right\} \tag{3.1}$$

which we now derive. On setting $\nu=n$ in Eq. (2.4) we have

$$\Gamma(n,x;z)=\sum_{m=0}^{n}\frac{(-z)^m}{m!}\Gamma(n-m,x)+\sum_{m=n+1}^{\infty}\frac{(-z)^m}{m!}\Gamma(n-m,x).$$

Next, by reversing the order of summation in the first sum and adjusting the initial index value of the second summation to zero we obtain after simplification

$$\Gamma(n,x;z)=\frac{(-z)^n}{n!}\left[\sum_{m=1}^{n}(-n)_m\left(\frac{1}{z}\right)^m\Gamma(m,x)+\sum_{m=0}^{\infty}\frac{(-z)^m}{(1+n)_m}\Gamma(-m,x)\right]. \tag{3.2}$$

The second summation in Eq. (3.2) is then evaluated with the aid of Eq. (2.2); thus, Eq. (3.1) is obtained.

By setting $n=0$ in Eq. (3.1) we see that

$$\Gamma(0,x;z)={}_0F_1[-;\ 1;\ z]\Gamma(0,x)-\frac{\mathrm{e}^{-x}}{x}\sum_{m=1}^{\infty}\frac{z^m}{(m!)^2}\sum_{k=0}^{m-1}k!\left(-\frac{1}{x}\right)^k \tag{3.3}$$

which (except for the omitted factor $\exp(-x)$) is given in [4, Theorem 13]. Since by employing series rearrangement

$$F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{——} : & \alpha, 1; & 1; \\ \mu, \nu : & \text{——}; & \text{—}; \end{matrix}\; xz,\ z\right] = \sum_{m=0}^{\infty} \frac{z^m}{(\mu)_m(\nu)_m} \sum_{k=0}^{m} (\alpha)_k x^k$$

$$= (\mu-1)(\nu-1) \sum_{m=1}^{\infty} \frac{z^{m-1}}{(\mu-1)_m(\nu-1)_m} \sum_{k=0}^{m-1} (\alpha)_k x^k, \tag{3.4}$$

we note that $\Gamma(n,x;z)$ given by Eq. (3.1) may be written in terms of a KdF function:

$$\Gamma(n,x;z) = \frac{(-z)^n}{n!}\left\{\sum_{k=1}^{n} \frac{(-n)_k}{z^k}\Gamma(k,x) + {}_0F_1[-;\ n+1;\ z]\Gamma(0,x)\right.$$

$$\left. - \frac{z}{n+1}\frac{e^{-x}}{x} F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———} : & 1,1; & 1; \\ 2, n+2 : & \text{——}; & \text{—}; \end{matrix}\; -x^{-1}z,\ z\right]\right\}. \tag{3.5}$$

4. Connection with incomplete Weber integrals

We recall that the general incomplete Weber integrals [10] are defined by

$$C_{e^2_{\mu,\nu}}(a,z) \equiv \int_0^z e^{at^2} t^{\mu} C_\nu(t)\,dt. \tag{4.1}$$

Here the subscript e^2 denotes the presence of the exponential function $\exp(at^2)$ in the integrand, the parameters μ, ν may be complex, and $C_\nu(t)$ is a cylindrical function. When $C_\nu(t) = I_\nu(t)$, then $I_{e^2_{\mu,\nu}}(a,z)$ converges provided that $\mathrm{Re}(1+\mu+\nu) > 0$; when $C_\nu(t) = K_\nu(t)$, then $K_{e^2_{\mu,\nu}}(a,z)$ converges provided that $\mathrm{Re}(1+\mu\pm\nu) > 0$.

Although the class of cylindrical functions $C_\nu(t)$ includes also Bessel functions of the first kind $J_\nu(t)$, Neumann functions $Y_\nu(t)$, and Hankel functions $H^{(1)}_\nu(t)$, $H^{(2)}_\nu(t)$ we shall employ below results for $C_{e^2_{\mu,\nu}}(a,z)$ pertaining only to the modified Bessel functions $I_\nu(t)$ and $K_\nu(t)$. Miller and Moskowitz have shown that $C_{e^2_{\mu,\nu}}\left(\frac{1}{4}a, z\right)$ is given by a linear combination of two KdF functions $F^{0:2;1}_{2:0;0}\left[a\frac{z^2}{4}, \xi\frac{z^2}{4}\right]$ where $\xi = 1$ if $C_\nu(z) = I_\nu(z), K_\nu(z)$ or $\xi = -1$ if $C_\nu(z) = J_\nu(z), Y_\nu(z), H^{(1)}_\nu(z), H^{(2)}_\nu(z)$. Thus by replacing the cylindrical function $C_\nu(z)$ by $I_\nu(z)$ and $K_\nu(z)$, respectively, in [10, Eq. (2.4)], we obtain two independent linear equations involving two KdF functions $F^{0:2;1}_{2:0;0}\left[a\frac{z^2}{4}, \frac{z^2}{4}\right]$. Noting that the determinant of this system is proportional to

$$I_\nu(z)K_{\nu-1}(z) + K_\nu(z)I_{\nu-1}(z) = \frac{1}{z},$$

we solve for the KdF functions thus obtaining

$$F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+1}{2};\ 1; \\ \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+3}{2} : \text{———};\ \text{—}; \end{matrix}\ a\frac{z^2}{4}, \frac{z^2}{4}\right]$$
$$= \frac{\mu-\nu+1}{z^\mu}\left\{K_{\nu-1}(z)I_{e^2_{\mu,\nu}}\left(\frac{a}{4},z\right) + I_{\nu-1}(z)K_{e^2_{\mu,\nu}}\left(\frac{a}{4},z\right)\right\} \tag{4.2a}$$

and

$$F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+1}{2};\ 1; \\ \dfrac{\mu+\nu+3}{2}, \dfrac{\mu-\nu+3}{2} : \text{———};\ \text{—}; \end{matrix}\ a\frac{z^2}{4}, \frac{z^2}{4}\right]$$
$$= \frac{(\mu+\nu+1)(\mu-\nu+1)}{z^{\mu+1}}\left\{I_\nu(z)K_{e^2_{\mu,\nu}}\left(\frac{a}{4},z\right) - K_\nu(z)I_{e^2_{\mu,\nu}}\left(\frac{a}{4},z\right)\right\}, \tag{4.2b}$$

where Re $(1+\mu\pm\nu)>0$. From these results we obtain, upon replacing μ by ν and ν by $1-\nu$,

$$F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———} : 1,\nu;\ 1; \\ 1, 1+\nu : \text{———};\ \text{—}; \end{matrix}\ -a\frac{z^2}{4}, \frac{z^2}{4}\right]$$
$$= \frac{2\nu}{z^\nu}\left\{I_{-\nu}(z)K_{e^2_{\nu,1-\nu}}\left(-\frac{a}{4},z\right) + K_{-\nu}(z)I_{e^2_{\nu,1-\nu}}\left(-\frac{a}{4},z\right)\right\} \tag{4.3a}$$

and

$$F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———} : 1,\nu;\ 1; \\ 2, 1+\nu : \text{———};\ \text{—}; \end{matrix}\ -a\frac{z^2}{4}, \frac{z^2}{4}\right]$$
$$= \frac{4\nu}{z^{\nu+1}}\left\{I_{1-\nu}(z)K_{e^2_{\nu,1-\nu}}\left(-\frac{a}{4},z\right) - K_{1-\nu}(z)I_{e^2_{\nu,1-\nu}}\left(-\frac{a}{4},z\right)\right\}, \tag{4.3b}$$

where Re $\nu>0$. Since $I_{-\nu}(z)=I_\nu(z)+(2/\pi)\sin(\pi\nu)K_\nu(z)$ Eqs. (4.3) may also be written as

$$F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———} : 1,\nu;\ 1; \\ 1, 1+\nu : \text{———};\ \text{—}; \end{matrix}\ -a\frac{z^2}{4}, \frac{z^2}{4}\right]$$
$$= \frac{2\nu}{z^\nu}\left\{I_\nu(z)K_{e^2_{\nu,\nu-1}}\left(-\frac{a}{4},z\right) + K_\nu(z)I_{e^2_{\nu,\nu-1}}\left(-\frac{a}{4},z\right)\right\} \tag{4.4a}$$

and

$$F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———} : 1,\nu;\ 1; \\ 2, 1+\nu : \text{———};\ \text{—}; \end{matrix}\ -a\frac{z^2}{4}, \frac{z^2}{4}\right]$$
$$= \frac{4\nu}{z^{\nu+1}}\left\{I_{\nu-1}(z)K_{e^2_{\nu,\nu-1}}\left(-\frac{a}{4},z\right) - K_{\nu-1}(z)I_{e^2_{\nu,\nu-1}}\left(-\frac{a}{4},z\right)\right\}, \tag{4.4b}$$

where Re $\nu>0$.

Hence by using the latter two results together with Eqs. (2.7) and (2.8) we have

$$\Gamma\left(-v,\frac{1}{x};\frac{z^2}{4}\right) = -\Gamma\left(1-v,\frac{1}{x}\right)\Gamma(v)\left(\frac{2}{z}\right)^v I_v(z)$$
$$+2\left(\frac{x}{z}\right)^v e^{-1/x}\left\{I_v(z)K_{e^2_{v,v-1}}\left(-\frac{x}{4},z\right)+K_v(z)I_{e^2_{v,v-1}}\left(-\frac{x}{4},z\right)\right\}, \tag{4.5}$$

where Re $v > 0$ and

$$\Gamma\left(-v,\frac{1}{x};\frac{z^2}{4}\right) = \Gamma\left(-v,\frac{1}{x}\right)\Gamma(1+v)\left(\frac{2}{z}\right)^v I_v(z)$$
$$-\left(\frac{x}{z}\right)^v x e^{-1/x}\left\{I_v(z)K_{e^2_{v+1,v}}\left(-\frac{x}{4},z\right)-K_v(z)I_{e^2_{v+1,v}}\left(-\frac{x}{4},z\right)\right\}, \tag{4.6}$$

where Re $v > -1$.

Finally, Eq. (4.6), for example, may be used together with Eq. (1.2) to obtain for Re $v > -1$

$$\Gamma\left(v,x;\frac{z^2}{4}\right) = 2\left(\frac{z}{2}\right)^v K_v(z) - \Gamma\left(-v,\frac{1}{x}\frac{z^2}{4}\right)\Gamma(1+v)\left(\frac{z}{2}\right)^v I_v(z)$$
$$+\frac{4x}{z^2}\left(\frac{x}{z}\right)^v e^{-\frac{1}{x}\frac{z^2}{4}}\left\{I_v(z)K_{e^2_{v+1,v}}\left(-\frac{x}{z^2},z\right)-K_v(z)I_{e^2_{v+1,v}}\left(-\frac{x}{z^2},z\right)\right\}. \tag{4.7}$$

Thus, we have shown that the generalized incomplete gamma function $\Gamma(v,x;z)$ for all complex v may be written essentially in terms of two incomplete Weber integrals. We need only then exploit known and other easily derived properties of the latter integrals in order to deduce properties of $\Gamma(v,x;z)$; this we shall do in the next Section.

We list here the following which are useful in computations pertaining to half odd integer orders of the generalized function $\Gamma(v,x;z)$:

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sinh(z), \qquad I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cosh(z),$$

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z},$$

$$\Gamma\left(\frac{1}{2},z\right) = \sqrt{\pi}\,\mathrm{erfc}(\sqrt{z}), \qquad \Gamma\left(-\tfrac{1}{2},z\right) = 2\left(\frac{e^{-z}}{\sqrt{z}} - \sqrt{\pi}\,\mathrm{erfc}\left(\sqrt{z}\right)\right).$$

5. Some properties of $I_{e^2_{v+1,v}}(a,z)$ and $K_{e^2_{v+1,v}}(a,z)$

Fortunately, many exact (and asymptotic) results pertaining to the subclasses $I_{e^2_{v+1,v}}(a,z)$ and $K_{e^2_{v+1,v}}(a,z)$ have already been recorded by Agrest and Maksimov in [2]. In addition to these, we list the following easily shown useful specializations:

$$I_{e^2_{1/2,-1/2}}\left(-\frac{a}{4},z\right) = \frac{1}{\sqrt{2a}}e^{1/a}\left[\mathrm{erf}\left(\frac{z\sqrt{a}}{2}+\frac{1}{\sqrt{a}}\right)+\mathrm{erf}\left(\frac{z\sqrt{a}}{2}-\frac{1}{\sqrt{a}}\right)\right] \tag{5.1a}$$

$$K_{e^2_{1/2,-1/2}}\left(-\frac{a}{4},z\right) = \frac{\pi}{\sqrt{2a}}\mathrm{e}^{1/a}\left[\mathrm{erf}\left(\frac{z\sqrt{a}}{2}+\frac{1}{\sqrt{a}}\right)-\mathrm{erf}\left(\frac{1}{\sqrt{a}}\right)\right]. \tag{5.1b}$$

Now setting $v=-\frac{1}{2},\frac{1}{2}$, respectively, in Eqs. (4.6) and (4.5), by using Eqs. (5.1) we deduce immediately

$$\Gamma\left(\frac{1}{2},x;z\right) = \frac{\sqrt{\pi}}{2}\left\{\mathrm{e}^{2\sqrt{z}}\mathrm{erfc}\left(\sqrt{x}+\sqrt{\frac{z}{x}}\right)+\mathrm{e}^{-2\sqrt{z}}\mathrm{erfc}\left(\sqrt{x}-\sqrt{\frac{z}{x}}\right)\right\} \tag{5.2a}$$

and

$$\Gamma\left(-\frac{1}{2},x;z\right) = \frac{1}{2}\sqrt{\frac{\pi}{z}}\left\{\mathrm{e}^{-2\sqrt{z}}\mathrm{erfc}\left(\sqrt{x}-\sqrt{\frac{z}{x}}\right)-\mathrm{e}^{2\sqrt{z}}\mathrm{erfc}\left(\sqrt{x}+\sqrt{\frac{z}{x}}\right)\right\}. \tag{5.2b}$$

Eq. (5.2a) has been recorded by W. Gautschi (see [1, Eq. (7.4.33)]) and Eqs. (5.2) have been derived by operational and direct methods in [3, 4].

Although the definition for the incomplete Weber integrals given by Eq. (4.1) is slightly different from the definitions given by Agrest and Maksimov in [2, p. 121], we may nonetheless use, respectively, the relations [2, p. 123, Eqs. (3.17) and (3.19)] along with the complete Weber integral for Re $a>0$, Re $v>-1$ (see e.g. [11, Section 2.16.8, Eq. (5)])

$$K_{e^2_{v+1,v}}(-a,\infty) = \frac{1}{2}\frac{\Gamma(v+1)}{(2a)^{v+1}}\mathrm{e}^{1/4a}\Gamma\left(-v,\frac{1}{4a}\right) \tag{5.3}$$

to deduce the relations

$$I_{e^2_{v+n+1,v+n}}(-a,z) = \left(\frac{1}{2a}\right)^n\left[I_{e^2_{v+1,v}}(-a,z)-\frac{z^v}{2a}\mathrm{e}^{-az^2}\sum_{k=1}^{n}(2az)^k I_{v+k}(z)\right] \tag{5.4a}$$

and

$$\begin{aligned} K_{e^2_{v+n+1,v+n}}(-a,z) = {} & \frac{1}{2}\frac{\mathrm{e}^{1/4a}}{(2a)^{v+n+1}}\left[\Gamma(v+n+1)\Gamma\left(-v-n,\frac{1}{4a}\right)-(-1)^n\Gamma(1+v)\Gamma\left(-v,\frac{1}{4a}\right)\right] \\ & +\left(-\frac{1}{2a}\right)^n\left[K_{e^2_{v+1,v}}(-a,z)-\frac{z^v}{2a}\mathrm{e}^{-az^2}\sum_{k=1}^{n}(-2az)^k K_{v+k}(z)\right], \end{aligned} \tag{5.4b}$$

where n is a nonnegative integer and Re $v>-1$. The condition Re $a>0$ has been waived in Eq. (5.4b) by appealing to the principle of analytic continuation. This is justified by noting that although $K_{e^2_{v+1,v}}(-a,\infty)$ exists only when Re $a>0$, the integral $K_{e^2_{v+1,v}}(-a,z)$ exists for any value of a as long as z is finite.

We now show that Eq. (5.4b) may be used to give an alternative derivation of Eq. (2.1a); thus, letting z approach zero we have

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{e}^{1/4a}}{(2a)^{v+n+1}}\left[\Gamma(v+n+1)\Gamma\left(-v-n,\frac{1}{4a}\right)-(-1)^n\Gamma(1+v)\Gamma\left(-v,\frac{1}{4a}\right)\right] \\ & -\left(-\frac{1}{2a}\right)^n\frac{1}{2a}\sum_{k=1}^{n}(-2a)^k\left[\lim_{z\to 0}z^{v+k}K_{v+k}(z)\right]=0. \end{aligned}$$

Since for $\operatorname{Re}\mu > 0$

$$\lim_{z\to 0} z^{\mu}K_{\mu}(z) = 2^{\mu-1}\Gamma(\mu) \tag{5.5}$$

we see that

$$\begin{aligned} &\mathrm{e}^{1/4a}\left[\Gamma(\nu+n+1)\Gamma\left(-\nu-n,\frac{1}{4a}\right)-(-1)^{n}\Gamma(1+\nu)\Gamma\left(-\nu,\frac{1}{4a}\right)\right] \\ &\quad =(-1)^{n}(4a)^{\nu}\sum_{k=1}^{n}(-4a)^{k}\Gamma(\nu+k). \end{aligned} \tag{5.6}$$

Now, setting $x = 1/4a$, replacing ν by $-\nu$, and waiving the condition $\operatorname{Re}(-\nu) > -1$ by invoking the principle of analytic continuation we obtain

$$\Gamma(\nu-n,x)=\frac{(-1)^{n}}{(1-\nu)_{n}}\left[\Gamma(\nu,x)-\frac{x^{\nu}\mathrm{e}^{-x}}{\nu}\sum_{k=1}^{n}(-\nu)_{k}\left(-\frac{1}{x}\right)^{k}\right] \tag{5.7}$$

which (on noting Eq. (2.3)) is equivalent to Eq. (2.1a).

Thus we also have on combining Eqs. (5.4b) and (5.6)

$$\begin{aligned} K_{e^{2}_{\nu+n+1,\nu+n}}(-a,z)= &\left(-\frac{1}{2a}\right)^{n}\left[K_{e^{2}_{\nu+1,\nu}}(-a,z)-\frac{z^{\nu}}{2a}\mathrm{e}^{-az^{2}}\sum_{k=1}^{n}(-2az)^{k}K_{\nu+k}(z)\right] \\ &-\left(-\frac{1}{2a}\right)^{n+1}2^{\nu-1}\sum_{k=1}^{n}\Gamma(\nu+k)(-4a)^{k}, \end{aligned} \tag{5.8}$$

where n is a nonnegative integer and $\operatorname{Re}\nu > -1$.

Finally, if we let $\nu = -\frac{1}{2}$ in Eqs. (5.4a) and (5.8), respectively, and use Eqs. (5.1), we obtain for $n = -1, 0, 1, \ldots$

$$\begin{aligned} I_{e^{2}_{n+\frac{3}{2},n+\frac{1}{2}}}\left(-\frac{a}{4},z\right)= &\left(\frac{2}{a}\right)^{n+\frac{1}{2}}\left\{\frac{\mathrm{e}^{1/a}}{a}\left[\operatorname{erf}\left(\frac{z\sqrt{a}}{2}+\frac{1}{\sqrt{a}}\right)+\operatorname{erf}\left(\frac{z\sqrt{a}}{2}-\frac{1}{\sqrt{a}}\right)\right]\right. \\ &\left.-\sqrt{\frac{2z}{a}}\mathrm{e}^{-(a/4)z^{2}}\sum_{k=0}^{n}\left(\frac{az}{2}\right)^{k}I_{k+\frac{1}{2}}(z)\right\} \end{aligned} \tag{5.9a}$$

and

$$\begin{aligned} K_{e^{2}_{n+3/2,n+1/2}}\left(-\frac{a}{4},z\right)= &(-1)^{n}\left(\frac{2}{a}\right)^{n+1}\left[\frac{\pi}{\sqrt{2a}}\mathrm{e}^{1/a}\operatorname{erfc}\left(\frac{z\sqrt{a}}{2}+\frac{1}{\sqrt{a}}\right)-\frac{\pi}{\sqrt{2a}}\mathrm{e}^{1/a}\operatorname{erfc}\left(\frac{1}{\sqrt{a}}\right)\right. \\ &\left.+\sqrt{\frac{\pi}{2}}\sum_{k=0}^{n}\left(\frac{1}{2}\right)_{k}(-a)^{k}-\sqrt{z}\mathrm{e}^{-(a/4)z^{2}}\sum_{k=0}^{n}\left(-\frac{az}{2}\right)^{k}K_{k+1/2}(z)\right]. \end{aligned} \tag{5.9b}$$

Although Eqs. (5.9) are incidental with regard to this study, we have recorded them since apparently not only are they new, but are the only known cases of incomplete Weber integrals containing modified Bessel functions (except for the specialized results given by Eqs. (5.1)) which are expressible in terms of well-known higher transcendental functions.

6. Representations for $\Gamma(\pm(\nu+n+1),x;z^2/4)$

Upon making the necessary substitutions and adjustments in the parameters of the generalized incomplete gamma function, we combine Eqs. (4.6), (4.7) with Eqs. (5.4) to obtain the following:

Theorem: *For* $\alpha=\nu+n+1$ $(n=-1,0,1,\ldots)$ *and* $\operatorname{Re}\nu>0$

$$\begin{aligned}\Gamma\left(\alpha,x;\frac{z^2}{4}\right) &= 2\left(\frac{z}{2}\right)^{\alpha}\Bigg\{K_\alpha(z)-\frac{1}{2}(-1)^n\Gamma(\nu)\Gamma\left(1-\nu,\frac{1}{x}\frac{z^2}{4}\right)I_\alpha(z)\\ &\quad-\left(\frac{2x}{z^2}\right)^{\nu}\mathrm{e}^{-\frac{1}{x}\frac{z^2}{4}}\left[K_\alpha(z)I_{e^2_{\nu,\nu-1}}\left(-\frac{x}{z^2},z\right)-(-1)^nI_\alpha(z)K_{e^2_{\nu,\nu-1}}\left(-\frac{x}{z^2},z\right)\right]\\ &\quad+\mathrm{e}^{-x-\frac{1}{x}\frac{z^2}{4}}\sum_{k=0}^{n}\left(\frac{2x}{z}\right)^{k+\nu}\left[K_\alpha(z)I_{\nu+k}(z)+(-1)^{n+k}I_\alpha(z)K_{\nu+k}(z)\right]\Bigg\}\end{aligned}\tag{6.1a}$$

and

$$\begin{aligned}\Gamma\left(-\alpha,x;\frac{z^2}{4}\right) &= 2\left(\frac{2}{z}\right)^{\alpha}\Bigg\{\frac{1}{2}(-1)^n\Gamma(\nu)\Gamma(1-\nu,x)I_\alpha(z)\\ &\quad+(2x)^{-\nu}\mathrm{e}^{-x}\left[K_\alpha(z)I_{e^2_{\nu,\nu-1}}\left(-\frac{1}{4x},z\right)-(-1)^nI_\alpha(z)K_{e^2_{\nu,\nu-1}}\left(-\frac{1}{4x},z\right)\right]\\ &\quad-\mathrm{e}^{-x-\frac{1}{x}\frac{z^2}{4}}\sum_{k=0}^{n}\left(\frac{z}{2x}\right)^{k+\nu}\left[K_\alpha(z)I_{\nu+k}(z)+(-1)^{n+k}I_\alpha(z)K_{\nu+k}(z)\right]\Bigg\}.\end{aligned}\tag{6.1b}$$

Thus knowing the values of $I_{e^2_{\nu,\nu-1}}(a,z)$ and $K_{e^2_{\nu,\nu-1}}(a,z)$ for $0<\operatorname{Re}\nu\leqslant 1$ is essentially sufficient to compute $\Gamma(\pm\alpha,x;z^2/4)$. We shall discuss the case $\Gamma(0,x;z)$ further in Section 7.

By letting z approach zero in Eq. (6.1a) and utilizing Eq. (5.5) we obtain Eq. (2.1b), since all the singularities are removable in the limit. It can be shown also that Eq. (6.1b) yields Eq. (2.1a), but the computation is somewhat delicate and not worth recording since it only provides a verification.

In particular, letting $\nu=\frac{1}{2}$ in Eqs. (6.1) and noting Eqs. (5.1) gives

$$\begin{aligned}\Gamma\left(\alpha,x;\frac{z^2}{4}\right) &= \left(\frac{z}{2}\right)^{\alpha}\Bigg\{K_\alpha(z)\operatorname{erfc}\left(\sqrt{x}-\frac{z}{2\sqrt{x}}\right)\\ &\quad+\left[K_\alpha(z)+(-1)^{\alpha-\frac{1}{2}}\pi I_\alpha(z)\right]\operatorname{erfc}\left(\sqrt{x}+\frac{z}{2\sqrt{x}}\right)\\ &\quad+2\mathrm{e}^{-x-\frac{1}{x}\frac{z^2}{4}}\sum_{k=0}^{\alpha-\frac{3}{2}}\left(\frac{2x}{z}\right)^{k+\frac{1}{2}}\left[K_\alpha(z)I_{k+\frac{1}{2}}(z)+(-1)^{\alpha+k+\frac{1}{2}}I_\alpha(z)K_{k+\frac{1}{2}}(z)\right]\Bigg\}\end{aligned}\tag{6.2a}$$

and

$$\Gamma\left(-\alpha,x;\frac{z^2}{4}\right)=\left(\frac{2}{z}\right)^{\alpha}\Bigg\{K_\alpha(z)\operatorname{erfc}\left(\sqrt{x}-\frac{z}{2\sqrt{x}}\right)$$

$$
- \left[K_\alpha(z) + (-1)^{\alpha-1/2}\pi I_\alpha(z)\right] \operatorname{erfc}\left(\sqrt{x} + \frac{z}{2\sqrt{x}}\right)
$$
$$
-2\mathrm{e}^{-x-\frac{1}{x}\frac{z^2}{4}} \sum_{k=0}^{\alpha-3/2} \left(\frac{z}{2x}\right)^{k+1/2} \left[K_\alpha(z) I_{k+1/2}(z) + (-1)^{\alpha+k+1/2} I_\alpha(z) K_{k+1/2}(z)\right] \Bigg\}, \tag{6.2b}
$$

where $\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. It is easy to verify that Eqs. (6.2) are valid a fortiori for $\alpha = -\frac{1}{2}$. Eq. (6.2a) has been obtained in [3, Eq. (6.1)] by finding independent solutions of certain recurrence relations derived from $\Gamma(\alpha, x; z)$.

7. Representations in terms of incomplete Lipschitz–Hankel integrals

Although we have already recorded formulas for the generalized incomplete gamma function of order zero (see Eqs. (1.3), (3.3)), this case deserves further attention since we may exhibit additional connections with other special functions, e.g. incomplete Lipschitz–Hankel integrals. In the general case, the incomplete Lipschitz–Hankel integrals [8] are defined by

$$
C_{e_{\mu,\nu}}(a,z) \equiv \int_0^z \mathrm{e}^{at} t^\mu C_\nu(t)\,\mathrm{d}t, \tag{7.1}
$$

where (except for the subscript e which denotes the presence of the exponential function $\exp(at)$ in the integrand) the functions $C_\nu(t)$, parameters, and convergence criteria are exactly the same as in the definition of the incomplete Weber integrals $C_{e^2_{\mu,\nu}}(a,z)$ represented by Eq. (4.1). The specialization $\mu = \nu$ in Eq. (7.1) is denoted by

$$
C_{e_\mu}(a,z) \equiv \int_0^z \mathrm{e}^{at} t^\mu C_\mu(t)\,\mathrm{d}t
$$

where for convergence of the integral at the lower limit of integration $\operatorname{Re}\mu > -\frac{1}{2}$ for either of the modified Bessel functions $I_\mu(t)$, $K_\mu(t)$.

Agrest and Maksimov have already noted certain connections between incomplete Weber integrals and incomplete Lipschitz–Hankel integrals (see [2, p. 135 et seq.]). When translated into the notation employed herein the pertinent results are

$$
I_{e^2_{1,0}}\left(-\frac{1}{4x}, z\right) = x\mathrm{e}^x \left[1 + \left(\frac{z}{4x} - \frac{x}{z}\right) I_{e_0}(-a,z) - \mathrm{e}^{-az} I_0(z)\right] \tag{7.2}
$$

and

$$
K_{e^2_{1,0}}\left(-\frac{1}{4x}, z\right) = x\mathrm{e}^x \left[\left(\frac{z}{4x} - \frac{x}{z}\right)\left(K_{e_0}(-a,z) + \frac{1}{2\sqrt{a^2-1}} \ln\left(\frac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}\right)\right) + \Gamma(0,x) - \mathrm{e}^{-az} K_0(z)\right], \tag{7.3}
$$

where

$$
a \equiv \frac{z}{4x} + \frac{x}{z}. \tag{7.4}
$$

Eq. (7.2) is obtained from [2, p. 136, Eq. (5.7)]. Eq. (7.3) is obtained from [2, p. 137, Eq. (5.12)] together with Eq. (5.3), but the condition Re $x > 0$ has been waived by appealing to the principle of analytic continuation.

Now, letting $\nu = 0$ and replacing x by $1/x$ in Eq. (4.6) we have

$$\Gamma\left(0,x;\frac{z^2}{4}\right) = \Gamma(0,x)I_0(z) - \frac{\mathrm{e}^{-x}}{x}\left[I_0(z)K_{e^2_{1,0}}\left(-\frac{1}{4x},z\right) - K_0(z)I_{e^2_{1,0}}\left(-\frac{1}{4x},z\right)\right]$$

which when used together with Eqs. (7.2) and (7.3) gives

$$\Gamma\left(0,x;\frac{z^2}{4}\right) = K_0(z) + \left(\frac{z}{4x} - \frac{x}{z}\right)$$
$$\times\left[K_0(z)I_{e_0}(-a,z) - I_0(z)K_{e_0}(-a,z) - \frac{1}{2}I_0(z)\frac{1}{\sqrt{a^2-1}}\ln\left(\frac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}\right)\right], \tag{7.5}$$

where a is given by Eq. (7.4). When $x = \frac{1}{2}z$, then $a = 1$ and Eq. (7.5) reduces to Eq. (1.3). A great deal is known about the incomplete Lipschitz–Hankel integrals appearing in Eq. (7.5). For example, there exist connections between $I_{e_0}(a,z)$, $K_{e_0}(a,z)$ and cylindrical functions of Poisson form (see [2, p. 137]) and the KdF functions $F^{0:2;1}_{2:1;0}\left[\frac{1}{4}a^2z^2, \frac{1}{4}z^2\right]$ (see [5–9]).

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